Schutz 7.4

$$\frac{d}{d\tau} p^0 + \Gamma^0_{\alpha\beta} p^\alpha p^\beta = 0.$$

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} g^{0\sigma} \left[g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\alpha} \right]$$

Demands $\sigma = 0$.

$$g_{\alpha\beta} = \begin{pmatrix} -(1+2\phi) & & & \\ & (1-2\phi) & & \\ & & (1-2\phi) & \\ & & & (1-2\phi) \end{pmatrix}$$

$\Rightarrow$ $g^{00} = \frac{-1}{1+2\phi}$.

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} \left[\frac{-1}{1+2\phi} \right] \left[g_{0\alpha,\beta} + g_{0\beta,\alpha} - g_{\alpha\beta,0} \right]$$

nonzero when $\alpha = \beta = 0$, or $\alpha = \beta = 1$

$\alpha = \beta = 0$:
$$\Gamma^0_{00} = \frac{1}{2} \left[\frac{-1}{1+2\phi} \right] \left[g_{00,0} \right]$$

$$= \frac{1}{2} \left[\frac{-1}{1+2\phi} \right] \left[-2\phi_{,0} \right]$$

$$= \frac{\phi_{,0}}{1+2\phi}$$

$$= \phi_{,0} \left[\frac{1}{1-(-2\phi)} \right]$$

$$= \phi_{,0} \left[\sum_{n=0}^{\infty} (-2\phi)^n \right]$$

$$= \phi_{,0} \left[1 - 2\phi + 4\phi^2 - 8\phi^3 + \cdots \right]$$

$$= \phi_{,0} - 2\phi_{,0}\phi + \cdots$$

$$= \boxed{\phi_{,0} + O(\phi^2)}$$

$\alpha = \beta = 1$: $\Gamma^0_{11} = \frac{1}{2}\left[\frac{-1}{1+2\phi}\right]\left[-g_{11,0}\right]$

By prompt, let g_{11} be $1 + O(\phi)$

$$= \frac{1}{2}\frac{-1}{1+2\phi}\, O(\phi_{,0})$$

$$= O(\phi_{,0})\left[1 - 2\phi + 4\phi^2 - \cdots\right]$$

$$= O(\phi_{,0}) + O(\phi^2).$$

This implies $\Gamma^0_{00} p^0 p^0$ and $\Gamma^0_{11} p^1 p^1$ are both $O(\phi_{,0})$ to leading order. In the nonrelativistic speed limit $p^0 \gg p^1$,

$$\boxed{\Gamma^0_{00} p^0 p^0 \gg \Gamma_{11} p^1 p^1}$$

$\Rightarrow$ g_{11} is irrelevant.

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Schutz 6.11 (a)

$$T^{\alpha}{}_{\beta;\nu} = T^{\alpha}{}_{\beta,\nu} + T^{\mu}{}_{\beta}\Gamma^{\alpha}_{\mu\nu} - T^{\alpha}{}_{\mu}\Gamma^{\mu}_{\beta\nu}$$

Replace $T^{\alpha}{}_{\beta}$ with $V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\mu\beta}V^{\mu}$

$$\Rightarrow V^{\alpha}{}_{;\beta\nu} = V^{\alpha}{}_{,\beta\nu} + (\Gamma^{\alpha}_{\mu\beta}V^{\mu})_{,\nu} + (V^{\mu}{}_{,\beta} + \Gamma^{\mu}_{\sigma\beta}V^{\sigma})\Gamma^{\alpha}_{\mu\nu} - (V^{\alpha}{}_{,\mu} + \Gamma^{\alpha}_{\sigma\mu}V^{\sigma})\Gamma^{\mu}_{\beta\nu}$$

$$V^{\alpha}{}_{;\nu\beta} = V^{\alpha}{}_{,\nu\beta} + (\Gamma^{\alpha}_{\mu\nu}V^{\mu})_{,\beta} + (V^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\sigma\nu}V^{\sigma})\Gamma^{\alpha}_{\mu\beta} - (V^{\alpha}{}_{,\mu} + \Gamma^{\alpha}_{\sigma\mu}V^{\sigma})\Gamma^{\mu}_{\beta\nu}.$$

Since $V^{\alpha}{}_{;\beta} = 0$, it's obvious that $V^{\alpha}{}_{;\beta\nu} - V^{\alpha}{}_{;\nu\beta} = 0$.

$$\Rightarrow 0 = V^{\alpha}{}_{;\beta\nu} - V^{\alpha}{}_{;\nu\beta}$$

$$= (\Gamma^{\alpha}_{\mu\beta}V^{\mu})_{,\nu} - (\Gamma^{\alpha}_{\mu\nu}V^{\mu})_{,\beta}$$

$$+ (V^{\mu}{}_{,\beta} + \Gamma^{\mu}_{\sigma\beta}V^{\sigma})\Gamma^{\alpha}_{\mu\nu} - (V^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\sigma\nu}V^{\sigma})\Gamma^{\alpha}_{\mu\beta}.$$

$$= \Gamma^{\alpha}_{\mu\beta,\nu}V^{\mu} + \cancel{\Gamma^{\alpha}_{\mu\beta}V^{\mu}{}_{,\nu}} - \Gamma^{\alpha}_{\mu\nu,\beta}V^{\mu} - \cancel{\Gamma^{\alpha}_{\mu\nu}V^{\mu}{}_{,\beta}}$$

$$+ \cancel{V^{\mu}{}_{,\beta}\Gamma^{\alpha}_{\mu\nu}} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\alpha}_{\mu\nu}V^{\sigma} - \cancel{V^{\mu}{}_{,\nu}\Gamma^{\alpha}_{\mu\beta}} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\alpha}_{\mu\beta}V^{\sigma}$$

$$= \boxed{(\Gamma^{\alpha}_{\mu\beta,\nu} - \Gamma^{\alpha}_{\mu\nu,\beta})V^{\mu} - (\Gamma^{\mu}_{\sigma\beta}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\mu}_{\sigma\nu}\Gamma^{\alpha}_{\mu\beta})V^{\sigma}}$$

(b) trivial

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Schutz 7.7

(a) Find as many conserved 4-momentum as possible.

(i) $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

By $m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta}\, p^\nu p^\alpha$

it's clear that $g_{\nu\alpha,\beta} = 0$, thus

all 4 of p_β are conserved.

(ii) $ds^2 = -(1-2M/r)dt^2 + (1-2M/r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

$$\Rightarrow g_{\alpha\beta} = \begin{pmatrix} -(1-\frac{2M}{r}) & & & \\ & \frac{1}{1-2M/r} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}$$

We use again $m\frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta}\, p^\nu p^\alpha$

It's clear that
$$\begin{cases} g_{\nu\alpha,\phi} = 0, & \Rightarrow \quad \frac{d}{d\tau} P_\phi = 0 \\ g_{\nu\alpha,t} = 0 & \Rightarrow \quad \frac{d}{d\tau} P_0 = 0 \end{cases}$$

(iii) $$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dt\,d\phi$$

$$+\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\,d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta$$

$$g_{\alpha\beta} = \begin{bmatrix} -\frac{\Delta - a^2\sin^2\theta}{\rho^2} & 0 & 0 & -2a\frac{2Mr\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -2a\frac{2Mr\sin^2\theta}{\rho^2} & 0 & 0 & \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta \end{bmatrix}$$

The identification of $g_{\nu\alpha,\phi} = 0$ is quick since ϕ does not appear in the metric $\Rightarrow \frac{d}{d\tau}p_\phi = 0$.

Also, t does not appear $\Rightarrow \frac{d}{d\tau}p_0 = 0$.

(IV) $ds^2 = -dt^2 + R^2(t)\left[(1-kr^2)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$

$$\Rightarrow g_{\alpha\beta} = \begin{bmatrix} -1 & \\ & R^2(t)\begin{bmatrix} (1-kr^2)^{-1} & \\ & r^2\begin{bmatrix} 1 & \\ & \sin^2\theta \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$\boxed{\frac{d}{d\tau} P_\phi = 0.}$$

(b) We know Cartesian → Spherical gives

$$g_{\alpha\beta} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \rightarrow g_{\alpha\beta} = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{bmatrix}$$

$\Rightarrow ds^2 = -t^2 + dx^2 + dy^2 + dz^2$ is equivalent to

$$g_{\alpha\beta} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix} \text{ in spherical.}$$

$$\Rightarrow ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad (i')$$

In this metric, which is spherical symmetric where the dependence of ds^2 on $d\theta, d\phi$ comes in the form of $\begin{pmatrix} r^2 & \\ & r^2\sin^2\theta \end{pmatrix}$.

we make the same identification on (iii) and (iv):

(iii): $g_{\alpha\beta} = \begin{pmatrix} \ddots & \\ & r^2\begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix} \end{pmatrix}$, (iv): $g_{\alpha\beta} = \begin{pmatrix} \ddots & \\ & R^2(t)\, r^2 \begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix} \end{pmatrix}$

This indeed implies p_θ is conserved.

(c). $\theta = \pi/2$, $p^\theta = 0$. solve p^r in m.

(i'): $g_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$

$\vec{p} \cdot \vec{p} = -m^2$

$\vec{p} \cdot \vec{p} = p^\alpha p^\beta g_{\alpha\beta}$

$= -(p^0)^2 + (p^r)^2 + (p^\theta)^2 + (p^\phi)^2$

$= -(p^0)^2 + (p^r)^2 + (p^\phi)^2 = -m^2$

$(p^r)^2 = -m^2 + (p^0)^2 + (p^\phi)^2$

$= E^2 - m^2 + (p^\phi)^2$

$\Rightarrow \boxed{p^r = \sqrt{E^2 - m^2 + (p^\phi)^2}}$

(ii):

$$g_{\alpha\beta} = \begin{pmatrix} -(1-2M/r) & & & \\ & (1-2M/r)^{-1} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}$$

$$\vec{p}\cdot\vec{p} = -m^2 = p^\alpha p^\beta g_{\alpha\beta}$$

$$= -(1-2\frac{M}{r})(p^0)^2 + \frac{(p^r)^2}{1-2M/r} + r^2(p^\theta)^2$$

$$+ r^2\sin^2\theta\,(p^\phi)^2$$

with $p^\theta = 0$, $(p^0)^2 = E^2$, we have

$$\frac{(p^r)^2}{1-2M/r} = -m^2 + E^2(1-\frac{2M}{r}) - r^2\sin^2\theta\,(p^\phi)^2$$

$$p^r = \sqrt{E^2 - (1-\frac{2M}{r})m^2 - (1-2\frac{M}{r})r^2\sin^2\theta\,(p^\phi)^2}$$

(iv): $-m^2 = -E^2 + R^2(t)\left[\frac{(p^r)^2}{1-kr^2} + r^2(p^\theta)^2 + r^2\sin^2\theta(p^\phi)^2\right]$

Multiply by $\frac{1-kr^2}{R^2}$:

$$(p^r)^2 = \left(\frac{1-kr^2}{R^2}\right)\left[E^2 - m^2\right] - r^2\sin^2\theta\,(p^\phi)^2$$

$$\Rightarrow \quad p^r = \sqrt{\left(\frac{1-kr^2}{R^2(t)}\right)\left[E^2 - m^2\right] - r^2\sin^2\theta\,(p^\phi)^2}$$

(d) when $k=0$, p_r conserved ?

$$k=0 \Rightarrow g_{\alpha\beta} = \begin{bmatrix} -1 & \\ & R^2(t) \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{bmatrix} \end{bmatrix}$$

$$(7.29): \quad m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta}\, p^\nu p^\alpha$$

$$\Rightarrow \quad m \frac{dp_r}{d\tau} = \frac{1}{2} g_{\nu\alpha,r}\, p^\nu p^\alpha$$

$g_{\nu\alpha,r}$ nonzero when $\begin{cases} \nu = \alpha = \theta, & g_{\theta\theta,r} = 2r R^2 \\ \nu = \alpha = \phi, & g_{\phi\phi,r} = 2r R^2 \sin^2\theta \end{cases}$

$$\Rightarrow \quad m \frac{dp_r}{d\tau} = \frac{1}{2} \left[g_{\theta\theta,r}\, p^\theta p^\theta + g_{\phi\phi,r}\, p^{\phi\phi} \right]$$

yet $p^\theta = p^\phi = 0$ by assumption, thus

$\boxed{p_r \text{ is conserved}}$

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